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A Lower Bound on the Variance of Conductance in Random Resistor Networks

Jan Wehr[1]

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We study the conductance of random resistor networks in $d \geqslant 2$ dimensions. It is shown (under some technical assumptions) that if a network exhibits a nonzero conductivity in the infinite-volume limit, then the variance of a finite-volume conductance grows at least like the volume.

KEY WORDS: Random media; conductivity; fluctuations; variance bounds.

In a real material, translational invariance is typically absent due to the presence of impurities and this effect often has to be taken into account for a proper understanding of the material's properties. Since positions of the impurities vary from one sample of the material to another, it is often assumed that they are distributed randomly. A way of accounting for this randomness mathematically is to make the structural parameters of the model random. Random resistor networks are mathematical models with random parameters, describing classical (as opposed to quantum) conductivity of disordered materials. More precisely, we consider a network of resistors with random conductivities and study its effective conductivity. In such a model, due to randomness, conductance (and its inverse—the resistance) is itself a random variable, as it depends on the random parameters of the model (strictly speaking, this statement is only true in a finite volume, i.e., for a finite network; see discussion of the infinite-volume limit below).

In this paper we address the question: how does the rate of conductance fluctuations depend on the volume of the network? We derive a lower bound which says that if the system exhibits a nonzero effective conductivity in the infinite-volume limit, the variance of the (appropriately

[1] Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

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normalized) finite-volume conductance is at least of the order of the volume. This is not an obvious result, since conductance depends on individual bond conductivities in a highly nontrivial way and is, in particular, far from being a sum of independent random variables. Similar results were obtained in ref. 8 for a class of extensive functions of independent random variables, that is, roughly speaking, for functions not far from being additive over the volume. We emphasize that conductance is not a quantity of this type and therefore different techniques have to be used to analyze its fluctuations. Our method is based on tools developed in ref. 1 for the purpose of studying fluctuations (more precisely, the moment generating function) of the difference of finite-volume free energies in the random-field Ising model (RFIM) with two different boundary conditions. This quantity was instrumental in proving absence of spontaneous magnetization in the two-dimensional RFIM (as well as of first-order phase transitions in some other random systems). The argument was by contradiction: had the spontaneous magnetization been nonzero, the fluctuations of the above-mentioned difference of free energies would be bigger than allowed by a simple uniform bound on the quantity. Thus the pivotal part of the argument can be roughly expressed as saying that a nonzero spontaneous magnetization drives the fluctuations of the free energy difference. The result of the present paper is, in a certain sense, analogous, and can be summarized by saying that a nonzero effective conductivity in the infinite-volume limit drives the fluctuations of the (finite-volume) conductance. Unlike the other statement, the present result is dimension independent.

Let us now introduce the two basic technical tools of the paper. In what follows the expectation of a random variable X will be denoted by $E[X]$. The variance of a random variable X is thus equal to

$$\mathrm{Var}[X]=E[X-E[X]]^2 \tag{1}$$

Lemma 1. Let $\eta_1,..., \eta_N$ be independent random variables on some probability space and let X be a function of N real variables, such that the random variable $X(\eta_1,..., \eta_N)$ has finite second moment. Let us introduce versions of this random variable averaged over all η_j except for the one with index i:

$$X_i=\int X(\eta_1,..., \eta_N) \prod_{j:\, j\neq i} \mu_j(d\eta_j) \tag{2}$$

Here μ_j denotes the distribution of η_j. Then

$$\mathrm{Var}[X]\geqslant \sum_{i=1}^{N} \mathrm{Var}[X_i] \tag{3}$$

Proof. The proof can be found in ref. 8.

Remark. Note that X_i is a function of η_i. In fact, it is simply the conditional expectation of $X(\eta_1,..., \eta_N)$, conditioned on the σ-field generated by the random variable η_i:

$$X_i = E[X(\eta_1,..., \eta_N) \mid \eta_i] \tag{4}$$

The lemma thus reduces the task of estimating the variance of a (possibly complicated) function of N independent variables η_i to that of estimating variances of functions of individual η_i.

Our second lemma shows how to estimate the variance of a function g of a real-valued random variable a, knowing that the expectation of $ag'(a)$ does not vanish. Other estimates of this type were used in ref. 1 to estimate fluctuations of free energy-like quantities.

Lemma 2. Suppose ν is a measure with a support in $[0, +\infty)$, having a differentiable density ρ with respect to the Lebesgue measure, such that

$$\int \frac{[\rho(a) + a\rho'(a)]^2}{\rho(a)} \, da < \infty \tag{5}$$

Define

$$\gamma_\nu(s) = \inf \left\{ \int g(a)^2 \, \nu(da) \mid g'(\cdot) \geqslant 0; \int ag'(a) \, \nu(da) = s \right\} \tag{6}$$

Then $\gamma_\nu(s) = 0$ only at $s = 0$.

Proof. Integrating by parts and using Schwarz' inequality, we get

$$\left| \int_0^{+\infty} ag'(a) \, \rho(a) \right| \leqslant \left[\int_0^{+\infty} g^2(a) \, \rho(a) \, da \right]^{1/2} \left[\int_0^{+\infty} \frac{[\rho(a) + a\rho'(a)]^2}{\rho(a)} \, da \right]^{1/2} \tag{7}$$

so that when $\int_0^{+\infty} ag'(a) \, \rho(a) \, da = s$ we get

$$\int_0^{+\infty} g^2(a) \, \rho(a) \, da \geqslant \frac{s}{\int_0^\infty \{[\rho(a) + a\rho'(a)]^2 / \rho(a)\} \, da} \tag{8}$$

and, by assumption, the last expression equals s times a positive constant. The proof is complete.

Remark. Condition (5) is equivalent to $\int_0^{+\infty} [\rho'(x)^2/\rho(x)]\, dx < \infty$. It is easy to see that this is satisfied in particular when $\rho(x) = \alpha e^{-\alpha x}$ (exponential distribution), $\rho(x) = (2/\sigma \sqrt{2\pi}) \exp(-x^2/2\sigma^2)$ (one-sided normal distribution), and by many other naturally occurring smooth densities, decaying sufficiently fast at $+\infty$.

In the simplest case considered here the conductivities of individual resistors are independent and identically distributed random variables. We assume for simplicity that the resistors are attached to the edges of the d-dimensional hypercubic lattice Z^d, although the result of this paper applies to many other situations. We thus have a family of i.i.d. random variables a_{xy}, one for each pair of integer points with Euclidean distance 1. Consider a finite part of the lattice—a cube Λ_L of size L, centered at the origin. The quantity we want to study is the conductance of the part of the network, contained in this cube, between its two opposite faces. The physical setup to be kept in mind is as follows: the opposite faces of the cube are connected to a battery which maintains a fixed potential difference. The potential is constant on these faces and for simplicity we will assume that

$$\phi_x = 0 \quad \text{for } x_1 = -L, \qquad \phi_x = 1 \quad \text{for } x_1 = L \tag{9}$$

The conductance is then given by Kelvin's variational principle:

$$\Sigma_L = \inf \sum_{|x-y|=1} a_{xy}(\phi_x - \phi_y)^2 \tag{10}$$

where the infimum is taken over all configurations of (real) potentials, ϕ_x, $x \in \Lambda_L$, subject to the above boundary conditions. The main goal of the paper is to study the fluctuations of Σ_L. Note that if the potential difference is chosen differently, the conductance is just multiplied by a constant, independent of the variables a_{xy}. We have normalized the potential difference so that Σ_L is of the order L^{d-2} (see below). The choice of the (Neumann) boundary conditions is made to fix the ideas: the results will not use it in any way. The crucial property of Σ_L that will be used in the proof is its homogeneity of degree 1 in the variables a_{xy}.

The main physical assumption we want to make is that the system exhibits bulk conductivity in the infinite-volume limit. Otherwise, with a predominant probability, $\Sigma_L = 0$ and the problem is not very interesting (except, of course, for the very difficult problem of the conductance fluctuations at the critical point; see ref. 6). A natural definition of the infinite-volume conductivity σ^* is

$$\sigma^* = \lim_{L \to \infty} \frac{\Sigma_L}{L^{d-2}} \tag{11}$$

Existence of the above limit is far from trivial, but it has been proven in a number of important cases. The (almost surely constant) limit is known to exist in the L^2 sense if the support of a_{xy} is bounded away from zero.[5,4] Hammersley[2] has observed that with special boundary conditions, almost sure existence of the limit follows for general distributions by the subadditive ergodic theorem. Recently Zhikov[10] has published a proof which applies to the Neumann boundary conditions in an analogous model defined in the continuum. Kesten[3] proved the existence of effective conductivity in the two-dimensional model with conductivities equal to 0 or 1 with probabilities $1-p$ and p respectively, for p close to 1. Finally, for hierarchical resistor networks, existence of the (almost surely constant) limit has been proven in L^2,[7] and in some cases almost surely.[9] In what follows we will for simplicity assume that the limit exists almost surely and that it is almost surely independent of the realization of the conductivities a_{xy}. It is easy to state and prove analogs of Theorem 1 with weaker assumptions, e.g., that the limit is nonzero and exists in probability.

Theorem 1. Suppose that the distribution measure ν of the conductivity variables satisfies the assumptions of Lemma 2, that the limit in (11) exists almost surely, and $\sigma^* > 0$. Then there exists a constant $C > 0$ such that

$$\liminf_{L \to \infty} L^{4-d} \operatorname{Var}[\Sigma_L] > 0 \tag{12}$$

Proof. It follows from the definition of Σ_L that it is a homogeneous function of degree 1 of the variables a_{xy}. By Euler's formula we have

$$\Sigma_L = \sum_{xy} a_{xy} \frac{\partial \Sigma_L}{\partial a_{xy}} \tag{13}$$

We will use this relation to estimate the variance of Σ_L. Let us apply the inequality of Lemma 1, which estimates the variance of a function of independent random variables by a sum of variances of its averaged versions. In our context the inequality reads

$$\operatorname{Var}[\Sigma_L] \geqslant \sum_{xy} \operatorname{Var}[\Sigma_{L,xy}] \tag{14}$$

where $\Sigma_{L,xy}$ is the result of integrating Σ_L over all $a_{x'y'}$ with $\{x, y\} \neq \{x', y'\}$:

$$\Sigma_{L,xy} = \int \Sigma_L \prod_{\{x',y'\} \neq \{x',y'\}} \nu(da_{x'y'}) \tag{15}$$

or, in probabilistic notation,

$$\Sigma_{L,xy} = E[\Sigma_L \mid a_{xy}] \tag{16}$$

(the conditional expectation of Σ_L, conditioned on the random variable a_{xy}). Thus $\Sigma_{L,xy}$ depends only on a_{xy}. In order to estimate the variance of $\Sigma_{L,xy}$ we will use Lemma 2. Now it follows from the definition of $\Sigma_{L,xy}$ that

$$\int_0^{+\infty} a_{xy} \frac{d\Sigma_{L,xy}}{da_{xy}} da_{xy} = \int \prod_{xy} \nu(da_{xy})\, a_{xy} \frac{d\Sigma_L}{da_{xy}} = E\left[a_{xy} \frac{d\Sigma_L}{da_{xy}} \right] \tag{17}$$

It follows from Lemma 2, together with the bound (14), that

$$\text{Var}[\Sigma_L] \geqslant \sum_{xy} \gamma\left(E\left[a_{xy} \frac{d\Sigma_L}{da_{xy}} \right] \right) \tag{18}$$

Now its definition implies that γ is a convex function, so that we obtain from (18)

$$\begin{aligned}\text{Var}[\Sigma_L] &\geqslant |\Lambda_L| \frac{1}{|\Lambda_L|} \sum_{xy} \gamma\left(E\left[a_{xy} \frac{d\Sigma_L}{da_{xy}} \right] \right) \\ &\geqslant |\Lambda| \gamma\left(\frac{1}{|\Lambda_L|} E\left[\sum_{xy} a_{xy} \frac{d\Sigma_L}{da_{xy}} \right] \right) = |\Lambda| \gamma\left(E\left[\frac{\Sigma_L}{|\Lambda_L|} \right] \right)\end{aligned} \tag{19}$$

Using the fact that γ is a homogeneous function of degree 2 (this follows easily from the definition), we can rewrite the last expression as

$$|\Lambda_L| \frac{L^{d-2^2}}{|\Lambda_L|} \gamma\left(E\left[\frac{\Sigma_L}{L^{d-2}} \right] \right) = cL^{d-4} \gamma\left(E\left[\frac{\Sigma_L}{L^{d-2}} \right] \right) \tag{20}$$

where c is a constant depending on the dimension only (we use here the fact that $|\Lambda_L|$ is to leading order equal to cL^d). By Fatou's lemma,

$$\liminf_{L \to \infty} E\left[\frac{\Sigma_L}{L^{d-2}} \right] \geqslant E\left[\liminf_{L \to \infty} \frac{\Sigma_L}{L^{d-2}} \right] = E[\sigma^*] = \sigma^* \tag{21}$$

so, by (obvious) monotonicity of γ,

$$\liminf_{L \to \infty} \frac{\text{Var}[\Sigma_L]}{L^{d-4}} \geqslant \gamma(\sigma^*) > 0 \tag{22}$$

which completes the proof.

Remark. We have only used the fact that the limit in (11) is not identically zero, and not that it is constant. However, the limit is expected to be constant in physically relevant situations.

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